Opinion Dynamics & Multistep Epigenetic Switch

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Advancement to Candidacy Examination

Motivations

- Economics (Sznajd-Weron, 2002)
- Politics (Ben-Naim, 2005)
- Physics (Ben-Naim et al., 2003)
- Sociology (Tavares, 2007)

Can we convince others to follow us?

- $n$ agents, with $n$ bounds of confidence $\{r_1, \ldots, r_n\}$
- agent $i$ has opinion $x_i(t) \in \mathbb{R}$
- $j$ is $i$'s out-neighbor if $|x_i(t) - x_j(t)| \leq r_i$

Hegselmann-Krause (HK) model

$$x(t + 1) = A(x(t), r)x(t)$$

$$a_{ij}(x(t), r) = \begin{cases} 
\frac{1}{\# \text{ of } i\text{'s neighbors}} & \text{if } j \text{ is } i\text{'s out-neighbor} \\
0 & \text{otherwise}
\end{cases}$$
Homogeneous HK
- converges in finite time
- the order of opinions is preserved
- in steady state agents are in agreement or disconnected

Heterogeneous HK
- may converge in infinite time
- pseudo-stable configurations
- disconnected clusters may reconnect

Conjectures
- when does convergence start?
- what is the final value?
- who moves and who stops?
- rate and direction of moving ones?
- do all heterogeneous HK’s converge?
Classification of Agents

- Open-minded
- Moderate-minded
- Closed-minded
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Conjectures

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Final value at constant topology

\[
x^*(x) = \lim_{t \to \infty} A(x, r)^t x = \begin{bmatrix} C & 0 & 0 \\ 0 & M & 0 \\ \Theta_C & \Theta_M & \Theta \end{bmatrix} x
\]

If \( A(x^*(x), r) = A(x, r) \), then
- \( x^*(x) \) is an equilibrium vector
- their proximity graph contains no moderate-minded

Rate of Convergence

Agent’s per-step convergence factor

\[
k_i(x(t)) = \frac{x_i(t+1) - x^*_i(x(t))}{x_i(t) - x^*_i(x(t))}
\]

\[k_i = 1 - \text{rate of convergence of agent } i\]

monotonic convergence toward final value \( \equiv 0 \leq k_i \leq 1 \)
Theorem (Evolution under Constant Topology after $\tau$)

- $x(t)$ converges to $x^*(x(\tau))$
- no moderate minded at $\tau$
- if $\rho(\Theta_\tau) \geq \rho(\Theta_{1,2,3})$, then for all $i \in G_{\Theta_\tau}$ and $j \in G_{\Theta_\tau}$
  
a) $\lim_{t \to \infty} k_i(x(t)) = \rho(\Theta_\tau)$
  
b) there exists $T \geq \tau$, after which $(x_i(t) - x_i^*)(x_j(t) - x_j^*) \geq 0$

In a Real Society

- the initial opinion of an open-minded has no effect, $x^* = \begin{bmatrix} C & 0 \\ \Theta_C & 0 \end{bmatrix} x$.
- an individual converges to his final decision as slow as the slowest group.
- the leader govern followers direction and rate.
- one can become a leader by joining a large strongly connected group.

Conjectures

- when does convergence start?
- what is the final value?
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Theorem (Sufficient Conditions for Convergence under Fixed Topology)

At each time step if for $x$ and $x^*(x)$ it holds that

1. $A(x, r) = A(x^*, r)$
2. if $x_i \leq x_j$, then $x_i^* \leq x_j^*$
3. $0 \leq k_i \leq 1$
4. for open-minded neighbors $i$ and $j$, $\Delta_i \Delta_j \geq 0$
5. for weakly connected open-mindeds $i$ and $j$

$$k_{\max,j} - k_{\min,j} \leq \min\{1 - k_{\max,j}, k_{\min,j}\} \min\{1 - \alpha \Delta_j, \beta \Delta_i\}$$

where $\Delta = x - x^*$, $n \in \mathbb{Z}^+$, $\alpha \in [k_{\min,j}, k_{\max,j}]$, and $\beta \in [k_{\min,i}, k_{\max,i}]$. 

\[ \Delta \]
Justification of Theorem Conditions

For any trajectory that converges under constant topology (1)

- $x$ converges to $x^* \rightarrow$ after some $\tau$ order is the same (2)
- $k_i$’s converge to spectral radii $\rightarrow 0 \leq k_i \leq 1$ (3)
- if $j$ is $i$’s successor, then after some $\tau$

$$\begin{cases} 
  \text{if } k_i \leq k_j \quad & \Delta_i \Delta_j \geq 0 \text{ and } k_i \rightarrow k_j \\
  \text{if } k_i \geq k_j \quad & \frac{\Delta_i}{\Delta_i} \rightarrow 0
\end{cases}$$

(4) and (5)

Conjectures

- when does convergence start?
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Future Work

- Convergence of heterogeneous HK
  1. what is the basin of attraction of $x^*$?
  2. what are possible $x^*$’s for a system?
  3. does the system fall into one of those basing of attractions?

Theorem (Convergence of products of stochastic matrices, Lorenz ’06)

$$\lim_{t \rightarrow \infty} A(t, 0) = \begin{bmatrix} C_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_m \end{bmatrix} A(0, 0)$$

- How can one become a leader by changing his $r$?

Future Work

- Disk Graphs
  - undirected disk graph
    homogeneous HK model
    $$\{i, j\} \in E \quad \text{if } |x_i - x_j| \leq r$$
  - directed disk graph
    heterogeneous HK model - confidence
    French model - influence
    $$\{i, j\} \in E \quad \text{if } |x_i - x_j| \leq r_i$$
    $$\{i, j\} \in E \quad \text{if } |x_i - x_j| \leq r_j$$
Motivations

Biology
- biofilm formation
- persistence
- adherence

Controls
- tristable
- phase variation
- epigenetic

protein production dynamics

phase varying dynamics

\[
\begin{align*}
\beta_{\text{on}} & : Y \xrightarrow{\alpha Y} \emptyset \\
\beta_{\text{partial}} & : Y \xrightarrow{\alpha Y} \emptyset \\
\beta_{\text{off}} & : Y \xrightarrow{\alpha Y} \emptyset
\end{align*}
\]
### Deterministic

\[ \dot{Y} = \beta - \alpha Y \quad Y_\infty = \frac{\beta}{\alpha} \]

### Stochastic

\[
P(t) = AP(t)
\]

\[
A = \begin{bmatrix}
-\beta & \alpha & 0 & 0 & \cdots \\
0 & -\beta & \alpha & 0 & \cdots \\
\beta & -\beta & -2\alpha & 3\alpha & 0 & \cdots \\
& \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

### Aggregation of CME

\[
P_{agg} = BP
\]

\[
B = \begin{bmatrix}
\frac{m_2}{m_1} & \cdots & \frac{m_2}{m_1} \\
\frac{m_3}{m_2} & \cdots & \frac{m_3}{m_1} \\
\alpha & \cdots & \alpha
\end{bmatrix}
\]

\[
P_{agg} = A_{agg}P_{agg}
\]

\[
A_{agg} = \begin{bmatrix}
-\frac{\beta}{m_1} & \frac{m_2}{m_1} & \frac{m_2}{m_2} & \frac{m_2}{m_3} & \alpha & \cdots \\
\frac{m_3}{m_1} & -\frac{m_3}{m_2} & \frac{m_3}{m_2} & \frac{m_3}{m_3} & \alpha & \cdots \\
\alpha & \cdots & \alpha
\end{bmatrix}
\]

\[
\beta(\text{protein number}) = \frac{\beta(\text{a.u.)}}{\Delta \text{fluorescence}}
\]
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**Lim’s Phase Varying Dynamics**

1. **Phase I**: \( \beta_{\text{on}} \) \( \xrightarrow{\alpha} Y \) \( \emptyset \)
2. **Phase III**: \( \beta_{\text{partial}} \) \( \xrightarrow{\alpha} Y \) \( \emptyset \)
3. **Phase V**: \( \beta_{\text{off}} \) \( \xrightarrow{\alpha} Y \) \( \emptyset \)

**Protein Production Dynamics**

- **Phase Varying Dynamics**

- **Stochastic Modeling of the Two Dynamics**

\[
\dot{P}(t) = AP(t)
\]
OxyR can not bind to hemimethylated
→ hemimethylated DNA plus OxyR is stable

Off gene after replication stays Off
→ OxyR goes off and gives naked gene in Partial state

gene undergoes conformational change
→ has not been observed

Future Work

- Finding the cheapest experiment to justify our model
  - plotting the histograms at each m generations
  - effect of changing the replication rate on histograms
  - appropriate repressor to compare possible models

- Design a controller based on this gene
  - control an unobserved model and still reach permanent Off
  - effect of phase varying rates on probability distribution
  - sensitivity analysis on On-Off switching rates